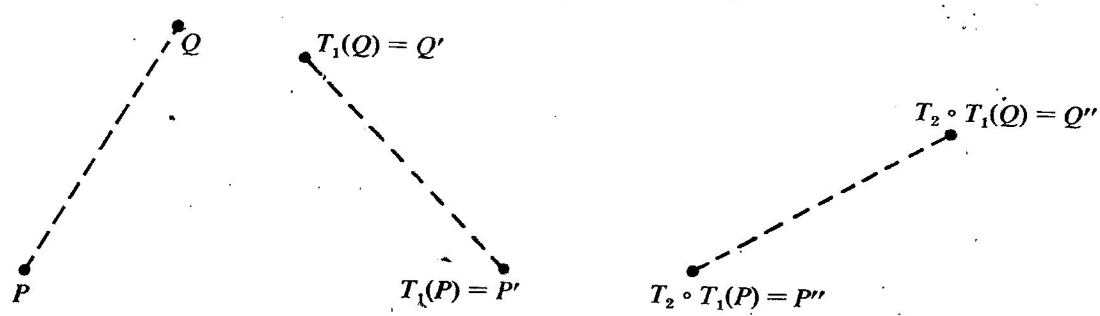
23.3 THE CONGRUENCE GROUP

When a set consists of all transformations which possess a given property (such as preserving distance), then with composition, the set often forms a group. Some of these groups are of particular interest.

Theorem 23.3.1 With composition, the set of all distance-preserving transformations forms a group.

Proof:

1. I preserves all properties of a figure. So I preserves distance.
2. If the distance between images equals the distance between preimages, then the reverse is true. So if a transformation preserves distance, then its inverse also does.
3. Suppose transformations T_1 and T_2 preserve distance. Let P and Q be points. Since T_1 preserves distance, $P'Q' = PQ$. Since T_2 preserves distance, $P''Q'' = P'Q'$. So $P''Q'' = PQ$ and the composite $T_2 \circ T_1$ preserves distance.



4. Composition of all transformations is associative. So composition of distance-preserving transformations is associative.

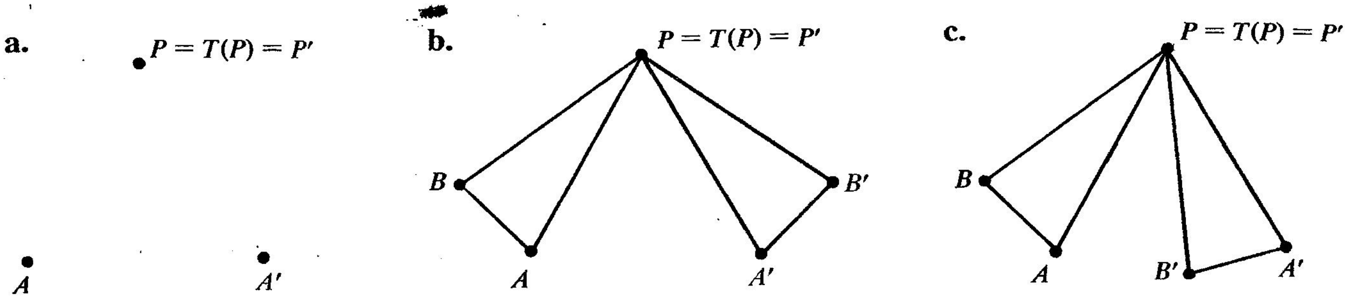
Because a reflection preserves distance, any isometry preserves distance. The question remains: Are there other kinds of distance-preserving transformations? We answer this question in the proof of the next theorem.

Theorem 23.3.2 Every distance-preserving transformation is an isometry.

The proof is divided into two cases. The first case is used in the proof of the second case.

Proof:

Case 1—Suppose distance-preserving transformation T fixes a point. That is, $T(P) = P$. Now consider another point A and suppose $T(A) = A'$. Since T preserves distance, $PA = PA'$. (See diagram **a** below.)



Now think of another point B. $T(B) = B'$. Since T preserves distance, $B'A' = BA$ and $PB' = PB$. Because of these restrictions, there are only two possible positions for B', shown in diagrams **b** and **c**. Diagram **b** shows a reflection over the perpendicular bisector of $\overline{AA'}$. Diagram **c** illustrates a rotation of magnitude $m\angle APA'$.

Thus, when there is a distance-preserving transformation T with a *fixed* point, the image of one point A determines the kind of transformation, and this transformation T is either a reflection or rotation. Thus T is certainly an isometry.

Case 2—Suppose T has no fixed point. That is, for any point P, $T(P) = P'$ and $P \neq P'$.

Let r be the reflection mapping P' onto P. $r(P') = P$. So,

$$r \circ T(P) = r(P') = P.$$

That is, $r \circ T$ is a distance-preserving transformation with a fixed point P. So, by Case 1, $r \circ T$ is an isometry. Certainly then, $r \circ r \circ T$ is an isometry. Finally, since $r \circ r = I$, T is an isometry.

Since each distance-preserving transformation is an isometry, and every isometry preserves distance:

The group of isometries with composition	is	The group of distance-preserving transformations with composition

This group is also called the **congruence group** because isometries may be used to define congruence.

If a set S is a subset of a second set T, and both S and T form groups with the same operation, then the group with S is said to be a **subgroup** of the group with T.

For example, with the operation addition,

$$E = \{\cdots, -4, -2, 0, 2, 4, \cdots\} = \text{the set of even integers}$$

forms a subgroup of the group of the set of all integers with addition.

Any symmetry group is a subgroup of the congruence group because (*1*) the operations are the same and (*2*) every symmetry set is a subset of the set of isometries. A special subgroup of the congruence group consists of all isometries which preserve orientation.

Theorem 23.3.3 With composition, the set of all orientation-preserving isometries forms a group.

The proof, similar to that of Theorem 23.3.1, is left as an exercise. Examples of orientation-preserving isometries are rotations and translations. Are there any others? This is answered in the next section.

EXERCISES

A

1. Name a transformation which does not preserve distance.

2. Because a reflection preserves distance, any composite of reflections ____.

3. Identify each set which forms a group with composition.

a. the set of isometries **b.** the set of distance-preserving transformations

c. the set of composites of reflections

4. What is the difference between the sets **a**, **b**, and **c** in Exercise 3?

5. What is the congruence group?

6. Name one subgroup of the congruence group.

7. The symmetry group of a rectangle is a subgroup of the congruence group. Why?

B

8. Suppose T is an isometry and r is a reflection. Why is $r \circ T$ also an isometry?

9. Name an isometry which has no fixed points.

Exercises 10-14 The given set and operation form a group. Name at least one subgroup.

10. $\{\cdots, -2, 0, 2, 4, 6, 8, \cdots\}$; addition

11. $\{I, r_{x\text{-axis}}, r_{y\text{-axis}}, R_{180}\}$; composition

12. $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\right\}$; matrix multiplication

C

13. the set of non-zero real numbers; multiplication

14. the set of orientation-preserving isometries; composition

15. Prove Theorem 23.3.3.

Exercises 16-18 Give arguments explaining why each set forms a group with composition.

16. the set of rotations with center (0,0)

17. the set of all translations

18. the set of isometries which map a given point P onto itself

23.4 IDENTIFYING ISOMETRIES

The congruence group is formed by the set of isometries and the operation composition. The set of isometries includes all reflections, rotations, translations, and glide reflections. (Refer to Chapter 12 if you have forgotten what a glide reflection is.) Does the set of isometries include any other transformations? In this section, we seek to describe all possible isometries—that is, all possible composites of reflections.

We begin with a surprising theorem about rotations. It states that many pairs of reflecting lines give rise to the same rotation.

Theorem 23.4.1

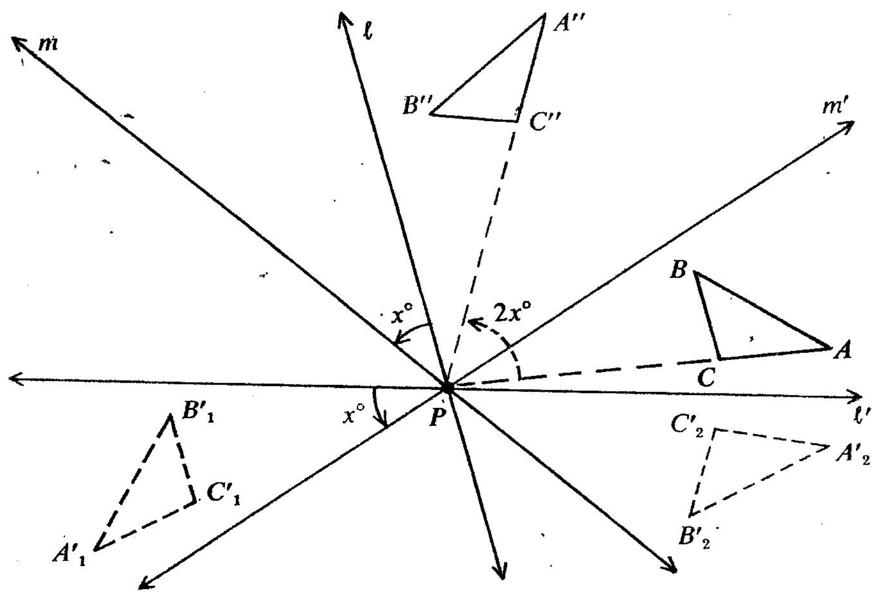
Suppose ℓ and m are lines intersecting at P. Let ℓ' and m' be the images of ℓ and m under any rotation with center P. Then

$$r_m \circ r_\ell = r_{m'} \circ r_{\ell'}.$$

Notice that, in the drawing,

$$\triangle A''B''C'' = r_m \circ r_\ell(\triangle ABC)$$

$$= r_{m'} \circ r_{\ell'}(\triangle ABC).$$



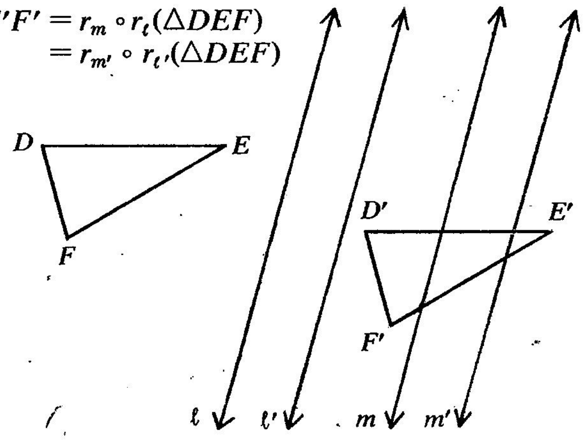
Proof: The proof is quite easy. The magnitude of rotation is twice the angle measure from ℓ to m. This measure is unchanged by rotating about P. So $r_m \circ r_\ell$ and $r_{m'} \circ r_{\ell'}$ have the same magnitude. Since they also have the same center, they must be the same rotation.

There is a corresponding theorem for translations.

Theorem 23.4.2 Suppose $\ell \parallel m$. Let ℓ' and m' be the images of ℓ and m under any translation. Then

$$r_m \circ r_\ell = r_{m'} \circ r_{\ell'}.$$

$\Delta D'E'F' = r_m \circ r_\ell(\Delta DEF)$
$= r_{m'} \circ r_{\ell'}(\Delta DEF)$



Proof: The magnitude of the translation is twice the distance from ℓ to m. The direction is perpendicular to ℓ and m. Each of these is unchanged by translating ℓ and m. (Check that this is true for the figure drawn at the left.)

In Case 1 of the proof of Theorem 23.3.2, it was shown that any isometry T which has a fixed point is either a reflection or a rotation. T is then the composite of at most 2 reflections. In Case 2 of the same proof, any *other* isometry was represented by $T = I \circ T = r \circ r \circ T$, and $r \circ T$ was an isometry with a fixed point. Since $r \circ T$ is then a composite of at most two reflections, $r \circ r \circ T = T$ is a composite of at most 3 reflections. So any isometry is the composite of at most 3 reflections.

Rotations and translations are the only composites of 2 reflections. Theorems 23.4.1 and 23.4.2 make it possible to describe all composites of 3 reflections.

Theorem 23.4.3 Let a, b, and c be any lines. The composite $r_c \circ r_b \circ r_a$ of three reflections is either a reflection or a glide reflection.

Proof: There are three cases, depending upon the positions of a, b, and c. (See diagrams at top of next page.)

In each of the first two cases,

$r_c \circ r_b \circ r_a = r_c \circ r_{b'} \circ r_{a'}$	(Theorems 23.4.1 and 23.4.2)
$= r_c \circ r_c \circ r_{a'}$	(since $c = b'$)
$= r_{a'}$	(since $r_c \circ r_c = I$)

So the composite of three reflections over parallel or concurrent lines is a reflection.

EXERCISES

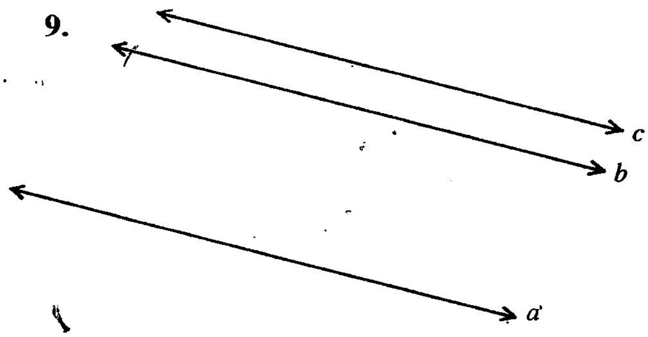
A 1. Name the four types of isometries.

Exercises 2-8 Name all types of isometries which:

2. preserve orientation.
3. reverse orientation.
4. are the composites of 2 reflections.
5. are the composites of 3 reflections.
6. are the composites of 4 reflections.
7. are the composites of 5 reflections.
8. are the composites of 10 reflections.

B **Exercises 9-10** Trace the lines a, b, and c. Find a line d so that $r_a \circ r_b = r_c \circ r_d$.

9.



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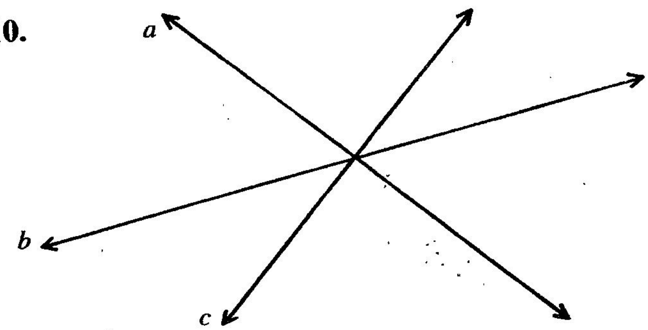
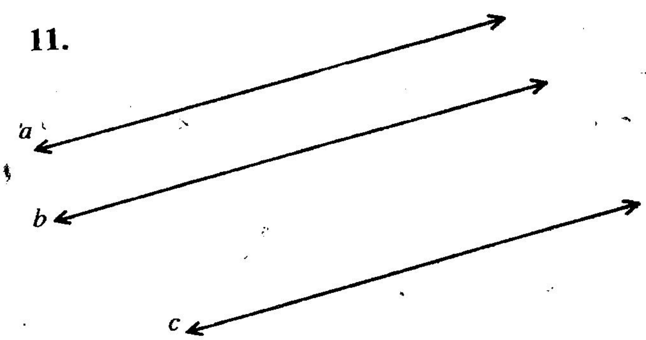
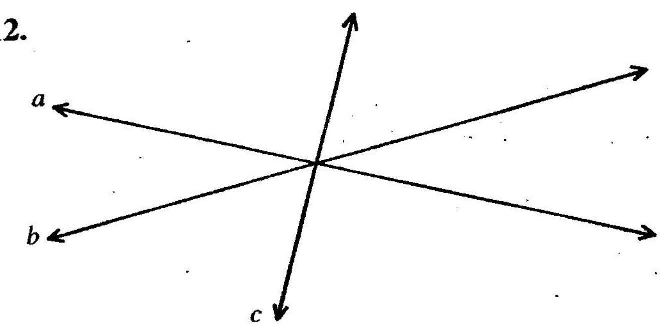


Exercises 11-12 Trace the lines a, b, and c. Find a line d so that $r_d = r_c \circ r_b \circ r_a$.

11.

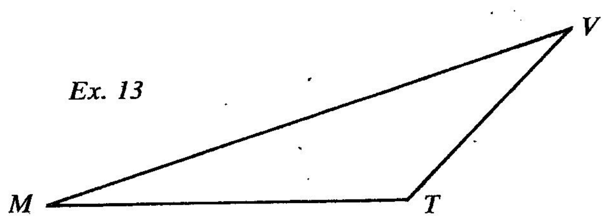


12.



13. Trace. Rotate $\triangle MTV$ 180° about T; then rotate the image 180° about M. The composite of these two rotations is a translation. What is the magnitude and direction of this translation?

Ex. 13



14. Draw two lines ℓ and m through a point P so that $r_\ell \circ r_m$ is a rotation of 100° about P.

15. Draw a segment $\overline{AB}$ of length 3″. Draw two lines p and q so that $r_p \circ r_q$ is a translation mapping A onto B.

16. What theorem of this section ensures that $r_{x\text{-axis}} \circ r_{x=y} = r_{x=y} \circ r_{y\text{-axis}}$?

Exercises 17-24 | Use any method to classify the isometry as one of the four types we have discussed.

17. $r_{x\text{-axis}} \circ r_{y=x}$

18. $r_{x\text{-axis}} \circ r_{x\text{-axis}} \circ r_{x\text{-axis}}$

Ⓒ 19. reflection over the x-axis followed by translation mapping (0,0) onto (−3,0)

20. reflection over the x-axis followed by rotation of 120° about (0,0)

21. reflection over the x-axis followed by rotation of 270° about (3,2)

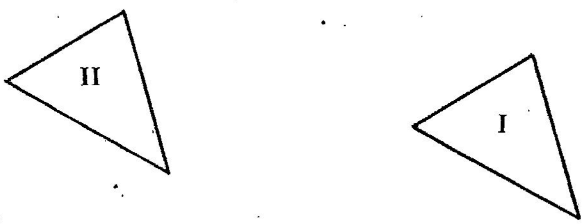
22. rotation of 40° about (1,1) followed by translation mapping (x,y) onto $(x - 1, y + 1)$

23. rotation of 60° about (2,4) followed by rotation of 300° about (6,1)

24. successive reflections over the three sides of the triangle $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$.

Exercises 25-28 | Which type of isometry seems to map triangle I onto triangle II? Copy each figure and determine the important points and/or lines of the isometry.

25.



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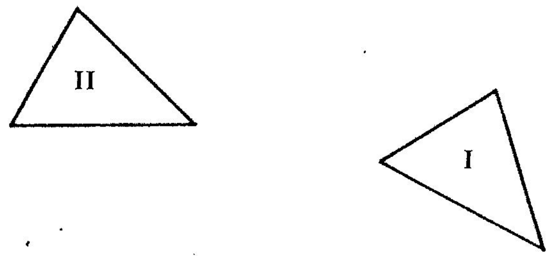

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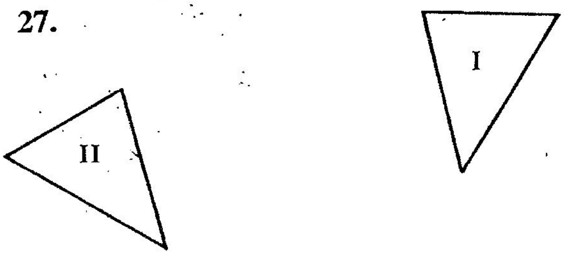

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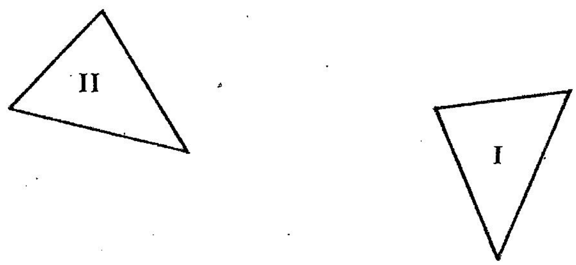

29. Let R_1 be the rotation of 240° about the origin and R_2 be the rotation of 120° about (6,2). Find the image of $\begin{bmatrix} 6 & 0 & 3 \\ 7 & 4 & 9 \end{bmatrix}$ under $R_2 \circ R_1$. Is $R_2 \circ R_1$ a rotation or translation?

30. Let S be the rotation of 90° about (1,1). Let T be the rotation of 200° about (7,3). Find the image of $\begin{bmatrix} 2 & 0 & -6 & 0 \\ 4 & 8 & 1 & -3 \end{bmatrix}$ under $T \circ S$. $T \circ S$ is a rotation. Approximate its center and magnitude.

31. Let R be a rotation of 40° about (0,0). Let T be the translation mapping (x,y) onto $(x, y + 4)$. Choose a triangle. Find its image under $R \circ T$. Describe $R \circ T$.

32. *Prove:* If the composite of two rotations of magnitudes a and b is a rotation, then the rotation has magnitude $a + b$.

33. *Prove:* If two rotations have magnitudes a and b, and $a + b = 360°$, then the composite of the two rotations is a translation.

23.5 THE SIMILARITY GROUP

The equivalence properties of congruence have counterparts in similarity.

1. $\alpha \sim \alpha$. The identity, being a size transformation, is a similarity transformation.
2. If $\alpha \sim \beta$, then $\beta \sim \alpha$. The inverse of a similarity transformation is also a similarity transformation (whose magnitude is the reciprocal of the magnitude of the given transformation).
3. If $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$. A similarity transformation is defined as a composite of isometries and size transformations. Consequently, the composite of the similarity transformations mapping α onto β and β onto γ is a similarity transformation mapping α onto γ.

Since composition is associative,

Theorem 23.5.1 With composition, the set of all similarity transformations forms a group.

This group is known as the **similarity group.** Since every isometry is a similarity transformation, the congruence group is a subgroup of the similarity group. Three other subgroups of the similarity group are particularly noteworthy.

Theorem 23.5.2 With composition, each of the following sets of similarity transformations forms a group:

a. the set of size transformations with a given point C as center;
b. the set of composites of size transformations and rotations with a given point C as center;
c. the set of transformations which map every line onto a line parallel to it.

The proof of each part of this theorem is left as an exercise. Group **a** is related to certain groups of real numbers and matrices. Group **b** is related to multiplication of numbers known as complex numbers. It is also related to certain figures known as spirals—the group is called the *spiral similarity group*. Group **c** is often used in the study of circles because the elements of the group map any circle onto any other circle. It consists of the size transformations, half-turns, and translations and is called the *homothety group*.

You may be wondering how many kinds of similarity transformations there are. Amazingly, the similarity transformations, like the isometries, can be classified as one of four types. Two of these types consist of transformations which preserve orientation:

1. Translations;

2. Spiral Similarities (composites of rotations and size transformations with the same point as center).

Two types reverse orientation:

3. Glide Reflections;

4. Reflective Similarities (composites of reflections and size transformations with the reflecting line containing the center of the size transformation).

These are the only similarity transformations.

EXERCISES

A

1. Describe the composite of a size transformation with center C, magnitude 3 and a size transformation with center C, magnitude 2.

2. Repeat Exercise 1 when the magnitudes are $\frac{2}{3}$ and $\frac{3}{5}$.

3. The identity size transformation has magnitude ____.

4. The inverse of a size transformation with center M, magnitude 10 is a ____.

5. By definition, a similarity transformation is ____.

6. Name two subgroups of the similarity group.

7. True or False? The similarity group is a subgroup of the congruence group.

B

Exercises 8-12 | Graph the image of $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & 3 \end{bmatrix}$ under each similarity transformation. Approximate coordinates where necessary.

8. the size transformation with center (3,2) magnitude 2

9. $r \circ s$, where r is the reflection over the x-axis and s is the size transformation with center (0,4), magnitude $\frac{4}{5}$

10. $s \circ r$, where r and s are as in Exercise 9

11. $s \circ t \circ u$, where u is the reflection over $y = -x$, t is the translation mapping (x,y) onto $(x+3, y+3)$, and s is the size transformation with magnitude 3, center (0,0) (Are the preimage and the image similar?)

12. $m \circ n$, where n is a size transformation with center $(2,-1)$, magnitude $\frac{1}{2}$, and m is a rotation with center (4,6), magnitude 180°

13. Make a diagram showing the relationship of each group to the others: congruence group, symmetry group of an equilateral triangle, a size transformation group, similarity group.

Ⓒ **14.** Make a diagram showing the relationship of each set to the others: isometries, reflections, identity, size transformations, similarity transformations, rotations, translations, glide reflections.

15. The composite of two size transformations S and T with centers A and B and magnitudes a and b is a size transformation. Find the center and magnitude of $T \circ S$ when $A = (0,0)$, $B = (10,0)$, $a = 2$, and $b = 3$.

16. Generalize Exercise 15 to cover any points and any magnitudes.

17. Let r be a reflection over a line m, and let s be a size transformation of magnitude k, center C. What is the inverse of $r \circ s$?

18. *Prove:* With multiplication, the set of all matrices of the form $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, $k > 0$, forms a group. How is this group related to this section?

23.6 WHAT IS GEOMETRY? (ANOTHER LOOK)

This question was first posed in Chapter 2. At that time a simple answer was given: Geometry is the study of figures. You can now understand a more sophisticated answer.

There are two major aspects to the study of geometry: the study of individual figures and the study of relationships among figures.

In the study of individual figures, symmetry plays an important role. And, in fact, knowing the symmetries of a figure enables you to know many of the properties of that figure. (Recall that one of the early theorems in this book stated that the base angles of an isosceles triangle are congruent, and this was proved easily because the isosceles triangle possesses symmetry.)

In the study of relationships among figures, the similarity transformations and the isometries are of utmost importance. They make possible the deter-